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Asymptotic solutions for two-dimensional low Reynolds number flow around an impulsively started circular cylinder

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The unsteady flow field of an incompressible viscous fluid around an impulsively started cylinder with slow motion is studied in detail. Integral expressions are derived from the nonlinear vorticity equation, and are solved by the method of matched asymptotic expansions. To complete the matching process five regions are necessary and their regions are essentially governed by the following relations: (i) the initial flow is unsteady Stokes flow (I), (ii) the early transient flow near the cylinder is steady Stokes flow (II), but the far-field flow is unsteady Stokes flow (III), so that Stokes–Oseen-like matching is necessary, and (iii) as time increases the inertia terms become significant far downstream; thus the far flow is unsteady Oseen flow (IV), but the flow near the cylinder is steady Stokes flow (V), so that the matching of the Stokes–Oseen equations is necessary. The asymptotic analytical solutions are given for five flow fields around a circular cylinder. Also presented are the drag coefficient, the vorticity, and the streamline. The drag coefficient is verified quantitatively by comparing with earlier theories of the initial flow and the steady flow. The streamline patterns calculated show the generation of a circulating zone close to the circular cylinder just as for the transient flow around a sphere, and the difference between two-dimensional and three-dimensional flows is discussed.

1. Introduction

The problem of unsteady viscous incompressible flow past a cylinder when a finite velocity is suddenly imparted to the cylinder is one of current interest in the full Navier–Stokes solutions. The analytic solution of the full Navier–Stokes equations for this flow field with finite Reynolds number is at present beyond our capabilities, but many numerical solutions have been reported (e.g. see Lecointe & Piquet 1984 for a representative bibliography). The analytic solution of the initial flow over an impulsively started circular cylinder at finite Reynolds numbers has been investigated by many investigators (e.g. Collins & Dennis 1973*a*, *b* and Badr & Dennis 1985). In these works, an iterative procedure is used, based on the assumption that the solution proceeds in power series of the normalized time. Bar-Lev & Yang (1975) develop this problem systematically, using the method of matched asymptotic expansions with respect to the time: the time-coordinate perturbation technique is applied and the region of non-uniformity near the cylinder is stretched to unity order.

The solution for the low Reynolds number flow past a solid sphere which represents the entire process of transition from stagnancy to the steady state has been studied by Bentwich & Miloh (1978) and Sano (1981). In their works, the Reynolds number based on the velocity which is suddenly imparted to the sphere in fluid initially at rest is assumed to be small and the method of matched asymptotic expansions is applied to the analysis. The time coordinate is differently scaled in the Stokes and Oseen domains. Bentwich & Miloh (1978) stated that an L-shaped region adjacent to the space and time axes is necessary in the matching process, but Sano (1981) pointed out that their L-shaped region is incomplete and corrected their matching procedure.

Whitehead's paradox occurs in the steady three-dimensional Stokes flow: the second approximation to the velocity of flow past a sphere remains finite at infinity in a way which is incompatible with the uniform stream condition, assuming an expansion of the flow in powers of the Reynolds number. In the two-dimensional Stokes flow, Stokes' paradox is stronger: the first approximation is incompatible. This fact implies that the unsteady low Reynolds number flow around a two-dimensional cylinder is in principle different from the three-dimensional flow. However, the two-dimensional transient low Reynolds number flow around an impulsively started cylinder has hitherto been unreported and the difference from the three-dimensional flow has not been discussed.

The present paper is an attempt to solve for the unsteady viscous incompressible flow past a two-dimensional cylinder which starts impulsively with a finite low rectilinear velocity, by using the method of matched asymptotic expansions. A new approach is devised to find the regions adjacent to the space and time axes in the matching process and to obtain the generalized inner and outer expansions, which are defined by Skinner (1975). Since the motion starts from rest, Laplace transform methods are used to account for the transient flow. The Oseen type of equation for vorticity is obtained from the full Navier–Stokes equations, and a system of integral equations whose kernels do not contain unknown functions is derived. Further they are simplified to integral equations with one variable, according to Kida & Take (1992*a*, *b*). These equations show that five regions are necessary to complete the matching process for time and space axes. The drag coefficient from stagnancy to the steady state is derived: it is singular at the beginning of motion and eventually reaches the value of the steady flow. The present result is verified by comparing with earlier results in both limiting cases, the beginning of the motion and the steady flow. The drag coefficient decreases monotonically with time, but the trend is slower than in the flow past a sphere. The streamlines of the entire flow field are presented and show that the features of the transient flow are almost the same as for the flow past a sphere: the circulating zone is generated in the vicinity of the surface of the cylinder. The major aim is to show the entire process of transition from stagnancy to the steady state in the two-dimensional flow and to discuss the difference between the two-dimensional flow and three-dimensional flow.

The method of matched asymptotic expansions used by Kaplun (1957) and Proudman & Pearson (1957) is a powerful method for finding uniformly valid asymptotic solutions of problems at low Reynolds number. Their asymptotic solutions are derived from the governing partial differential equations, that is, the Navier–Stokes equations. In their approach, careful consideration is necessary to complete the matching process, as pointed out by Kida & Miyai (1978*a*, *b*) and Sano (1981). An alternative method derived from integral equations was proposed by Kida & Miyai (1973). This method was used by Guermond (1987, 1990) and Guermond & Sellier (1991), in which they pointed out that this method is systematic. Kida & Miyai (1978*a*, *b*) show that we can derive all terms which are necessary in the matching systematically. A rigorous

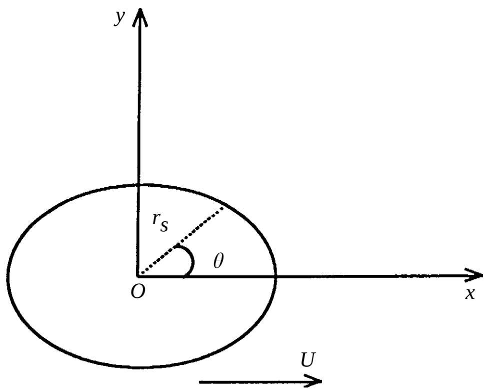

FIGURE 1. Coordinate systems and physical state.

analysis of this method is reported by Kida (1991). Therefore, this method from the integral equations is believed to be more powerful than that of Kaplun (1957) and Proudman & Pearson (1957) in this problem, because the present problem is more complicated than the three-dimensional problem, as mentioned above. Full details of the analysis are contained in a typescript in the editorial files of the Journal of Fluid Mechanics and a copy may be obtained on request to the first-named author or to the Editor.

2. Basic relations

2.1. *Integral formulation*

Figure 1 shows a cylinder which impulsively starts with slow velocity U. The x-axis is parallel to the velocity U and the y-axis is upward. The origin O is set at a typical point of the cylinder at a time t. The surface of the cylinder is denoted by the polar coordinates $(r_s(\theta_s), \theta_s)$. The Navier–Stokes equations for the two-dimensional unsteady motion of incompressible viscous fluid are given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\operatorname{grad} P + \boldsymbol{K} + \nu \nabla^2 \boldsymbol{u}, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2}$$

where $\boldsymbol{u}$ is the velocity vector, P is the pressure divided by the density of the fluid, ν is the kinematic viscosity and $\boldsymbol{K}$ is the mass force. Velocities and lengths are normalized with U and the typical length of the cylinder d. For a circular cylinder, we will take d to be the radius of the circular cylinder. From these equations (1) and (2) the governing vorticity equations in dimensionless form are

$$\left.\begin{aligned} \frac{\mathrm{D}\zeta}{\mathrm{D}t} &= \frac{1}{Re}\nabla^2\zeta, \\ \zeta &= -\nabla^2\Psi, \end{aligned}\right\} \tag{3}$$

where $\mathrm{D}/\mathrm{D}t$ is the substantial derivative, ζ is dimensionless vorticity, Ψ is the stream function, t is dimensionless time, and $Re = Ud/\nu$ is Reynolds number. The mass force $\boldsymbol{K}$ is assumed to be conservative, so that it does not affect the flow kinematics.

Here let ψ be the perturbation stream function. Then, the vorticity ζ is given by

$$\zeta = -\nabla^2\psi. \tag{4}$$

The boundary and initial conditions of the unsteady flow around an impulsively started body are given by

$$\left.\begin{array}{ll} u, v, \zeta \to 0 & \text{where } |\boldsymbol{x}| \to \infty, \\ v = 0, \quad u = H(t) & \text{on } \; S, \end{array}\right\} \tag{5}$$

where u and v are dimensionless perturbation velocity components in the x- and y-directions respectively, S is the boundary surface of the body, and $H(t)$ is the Heaviside step function; $H(t)$ is unity for $t > 0$ and zero otherwise. The first condition of (5) implies that the cylinder is started impulsively from rest and the second condition is that the cylinder subsequently moves with constant velocity.

Equation (3) is rewritten as

$$\frac{1}{Re}\nabla^2\zeta - \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} = f(x, y, t) \tag{6}$$

where

$$f(x, y, t) = u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}. \tag{7}$$

The Laplace transform method is used to account for the transient flow. We let $\tilde{\zeta}$ and $\tilde{f}$ be the Laplace transforms of ζ and f, respectively:

$$\tilde{\zeta} = \int_0^\infty \exp(-pt)\zeta\,\mathrm{d}t, \tag{8}$$

$$\tilde{f} = \int_0^\infty \exp(-pt) f\,\mathrm{d}t. \tag{9}$$

Then, $\tilde{\zeta}$ is governed by the following relation in view of the initial condition (5) ($\zeta = 0$ at $t = 0$):

$$\frac{1}{Re}\nabla^2\tilde{\zeta} - p\tilde{\zeta} + \frac{\partial \tilde{\zeta}}{\partial x} = \tilde{f}. \tag{10}$$

We define ε for simplicity of notation as

$$\varepsilon \equiv \tfrac{1}{2}Re \tag{11}$$

Further, we define $\hat{\zeta}$ as

$$\hat{\zeta} \equiv \exp(\varepsilon x)\tilde{\zeta}. \tag{12}$$

Then, (10) becomes

$$\nabla^2\hat{\zeta} - \varepsilon(\varepsilon + 2p)\hat{\zeta} = 2\varepsilon\tilde{f}\exp(\varepsilon x). \tag{13}$$

From the Gauss divergence theorem, the solution of (13) is obtained as

$$\hat{\zeta}(\boldsymbol{x}_o) = -\frac{\varepsilon}{2\pi}\int_D G(\boldsymbol{x}, \boldsymbol{x}_o)F(\boldsymbol{x})\mathrm{d}x\mathrm{d}y + \frac{1}{2\pi}\oint_S \left[G(\boldsymbol{x}_s, \boldsymbol{x}_o)\frac{\partial \hat{\zeta}(\boldsymbol{x}_s)}{\partial n} - \hat{\zeta}(\boldsymbol{x}_s)\frac{\partial G(\boldsymbol{x}_s, \boldsymbol{x}_o)}{\partial n}\right]\mathrm{d}s \tag{14}$$

where $\boldsymbol{x} = (x, y)$, $\boldsymbol{x}_o = (x_o, y_o)$ is in D which is the entire region outside the cylinder, $\boldsymbol{x}_s$ is the vector on the surface S of the cylinder, ds is a small length on S, and n is the normal to the surface of the cylinder, outward with respect to D. The function F

is defined by

$$F = 2\exp(\varepsilon x)\tilde{f}, \tag{15}$$

and the fundamental solution $G(\boldsymbol{x}, \boldsymbol{x}_o)$ is governed by the following equation:

$$\nabla^2 G - \varepsilon(\varepsilon + 2p)G = -2\pi\delta(\boldsymbol{x} - \boldsymbol{x}_o) \tag{16}$$

where $\delta(\boldsymbol{x})$ is the two-dimensional Dirac delta function. The solution G is easily found to be given by

$$G(\boldsymbol{x}, \boldsymbol{x}_o) = K_0(a\rho_r)$$

where $\rho_r = |\boldsymbol{x} - \boldsymbol{x}_o|$, K_0 is the zeroth-order modified Bessel function of the second kind and

$$a^2 = \varepsilon(\varepsilon + 2p). \tag{17}$$

We let $\tilde{\psi}$ be the Laplace transform of ψ:

$$\tilde{\psi} = \int_0^\infty \exp(-pt)\psi \mathrm{d}t. \tag{18}$$

Then $\tilde{\psi}$ is governed, from (4), by

$$\tilde{\zeta} = -\nabla^2\tilde{\psi}. \tag{19}$$

From the boundary and initial conditions (5), we have the following relations:

$$\left.\begin{aligned} &\tilde{\psi} = \frac{y}{p}, \quad \frac{\partial\tilde{\psi}}{\partial n} = \frac{1}{p}\frac{\partial y}{\partial n} && \text{on } S, \\ &\frac{\partial\tilde{\psi}}{\partial x}, \quad \frac{\partial\tilde{\psi}}{\partial y} \to 0 && \text{as } |\boldsymbol{x}| \to \infty, \\ &\tilde{\zeta} \to 0 && \text{as } |\boldsymbol{x}| \to \infty. \end{aligned}\right\} \tag{20}$$

Taking into account (20), we can easily obtain $\tilde{\psi}$ as

$$\tilde{\psi}(\boldsymbol{x}_o) = \frac{1}{2\pi}\int_D \tilde{\zeta}\log\left(\frac{1}{\rho_r}\right)\mathrm{d}x\mathrm{d}y. \tag{21}$$

The additional term $(1/2\pi)\oint_S(-\log\rho_r(\partial\tilde{\psi}/\partial n) + \tilde{\psi}(\partial\log\rho_r/\partial n))\mathrm{d}s$ must be zero, because this term is a potential flow with the boundary conditions $v = 0$ on S and u, $v \to 0$ as $|\boldsymbol{x}| \to \infty$. Details of the derivation of G and (21) are in the typescript in the editorial files.

2.2. *Expressions with respect to one variable*

We here assume that the cylinder is symmetric with respect to the x-axis. Then we can reduce (14) and (21) to simple integral expressions, following the idea of Kida & Take (1992*a*, *b*). Equation (14) can be expressed as

$$\hat{\zeta}(r, \varphi) = -\frac{\varepsilon}{2\pi}\int_0^{2\pi}\int_{r_s(\theta)}^\infty G(r, \varphi, r_1, \theta)F(r_1, \theta)r_1\mathrm{d}r_1\mathrm{d}\theta + F_0(r, \varphi).$$

where $\boldsymbol{x} = (x, y) = (r_1\cos\theta, r_1\sin\theta)$, $\boldsymbol{x}_o = (x_o, y_o) = (r\cos\varphi, r\sin\varphi)$, and $\boldsymbol{x}_s = (x_s, y_s) = (r_s(\theta_s)\cos\theta_s, r_s(\theta_s)\sin\theta_s)$. The function F_0 is defined by

$$F_0(r, \varphi) \equiv \frac{1}{2\pi}\oint_S \left[G\frac{\partial\hat{\zeta}}{\partial n} - \hat{\zeta}\frac{\partial G}{\partial n}\right]\mathrm{d}s. \tag{22}$$

The above relation can also be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\hat{\zeta}(r,\varphi)+\varepsilon\int_{r_s(\theta)}^{\infty}G(r,\varphi,r_1,\theta)F(r_1,\theta)r_1\mathrm{d}r_1-F_0(r,\varphi)\right]\mathrm{d}\theta=0.$$

Let f^s be some odd function with respect to θ, that is, $\int_0^{2\pi} f^s\mathrm{d}\theta = 0$. Then we can reduce (14) to a simple expression:

$$\hat{\zeta}(r,\varphi)=-\varepsilon\int_{r_s(\theta)}^{\infty}F(r_1,\theta)G(r,\varphi,r_1,\theta)r_1\mathrm{d}r_1+F_0(r,\varphi)+f^s(r,\theta,\varphi). \quad (23)$$

Similarly (21) is reduced to

$$\tilde{\psi}(r,\varphi)=\int_{r_s(\theta)}^{\infty}\exp\left(-\varepsilon r_1\cos\theta\right)\hat{\zeta}(r_1,\theta)\log\left(\frac{1}{\rho_r}\right)r_1\mathrm{d}r_1+g^s(r,\theta,\varphi) \quad (24)$$

where $\rho_r=(r^2+r_1^2-2rr_1\cos(\theta-\varphi))^{1/2}$. Here g^s is defined as some odd function with respect to θ. The system of equations (23) and (24) is considered to be the integral equations with respect to the variable r with a parameter θ. The idea of Kida (1991) and Kida & Take (1995) is, therefore, applicable to these equations.

3. Asymptotic analysis

3.1. *Local regions*

We consider asymptotic solutions of the integral equations (23) and (24) with the boundary conditions (20) with respect to the small parameter ε. The equations are nonlinear on $\tilde{\psi}$ and $\hat{\zeta}$, but these unknown functions are not contained in the kernel functions of (23) and (24). Therefore, the analysis given by Kida (1991) and Kida & Take (1995) will apply to the present problem.

We assume that the solutions are sufficiently differentiable except at the boundary of the cylinder and their regular and local expansions exist in a subdomain of the entire region outside the cylinder. Since the constant a defined by (17) is of the order of the function ε, $a=\{\varepsilon(\varepsilon+2p)\}^{1/2}$, we let a be a small perturbation parameter. We define the integral operator $\mathscr{K}_a$:

$$\mathscr{K}_a\phi=\int_{r_s(\theta)}^{\infty}K_0(a\rho_r)\phi(r_1)\mathrm{d}r_1 \quad (25)$$

where ϕ is an arbitrary test function independent of a, for which the above integral exists. Following Kida (1991), the integral operator $\mathscr{K}_a$ is decomposed as

$$\mathscr{K}_a=\mathscr{K}_0+\mathscr{K}_p \quad (26)$$

where

$$\mathscr{K}_0\phi=\int_{r_s(\theta)}^{\infty}\left(-\gamma-\log\left(\tfrac{1}{2}\rho_r\right)-\log a\right)\phi(r_1)\mathrm{d}r_1. \quad (27)$$

Here γ is Euler's constant and the operator $\mathscr{K}_p$ satisfies

$$\lim_{a\to 0}\mathscr{K}_p\phi=0.$$

In deriving the above relation, we have used the asymptotic expansion of the modified Bessel function:

$$K_0(\rho_r)\sim-\log\left(\tfrac{1}{2}\rho_r\right)-\gamma \quad \text{for} \quad \rho_r\ll 1. \quad (28)$$

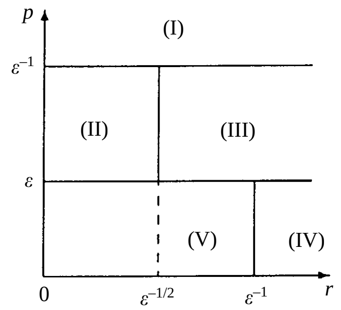

FIGURE 2. Schematic diagram of the matching regions.

We have to note that the logarithmic term, $\log a$, is included in the kernel of the operator $\mathscr{K}_0$, that is, the generalized expansions will be used in the present analysis.

The region near $\boldsymbol{x} = \infty$ is considered to be significant if a significant local region exists. We define the local operator $\mathscr{K}^*_a$ which is given by the local variable $r^* = \delta r$ where $\delta \to 0$ as $a \to 0$:

$$\mathscr{K}^*_a \phi^* = \int_{\delta r_s(\theta)}^{\infty} K_0\left(\frac{a}{\delta}\rho^*_r\right) \phi^*(r^*_1)\mathrm{d}r^*_1 \tag{29}$$

where ϕ^* is a test function independent of a and $\rho^*_r = (r^{*2} + r_1^{*2} - 2r^* r^*_1 \cos(\theta - \varphi))^{1/2}$. If we take δ to be a, we easily see that the above local operator $\mathscr{K}^*_a$ is significant (the definition of 'significant' is given in Kida 1991). From this fact, the significant local variable is given by $r^* = ar$.

Thus, the significant local region of the present problem is dependent on the order of p, since a is dependent on p and ε. Figure 2 shows a schematic diagram of the local regions. When p is of the order of $1/\varepsilon$, that is region (I), there is no significant local region, since a is of the order of unity. When p is of the order of unity, the significant local region (III) exists for $r = O(1/\varepsilon^{1/2})$. When p is of the order of ε, the significant local region (IV) exists for $r = O(1/\varepsilon)$.

A similar discussion is necessary for (24). We easily see that the significant local variable is given by $R = \varepsilon r$ independent of the order of p. As will be shown in the following sections, 3.2 and 3.3, $\hat{\zeta}$ is exponentially small for $R = O(1)$ in the cases of $p = O(1)$ and $O(1/\varepsilon)$. Therefore, the significant local region of this equation is contained in the above local regions.

Note that there is the possibility of the existence of other local regions arising from the nonlinear term F. For steady flow at low Reynolds number, this possibility is zero (Kida & Take 1992*a*), so it also may be zero for the present problem. Further, we have to note that the above discussion shows the possibility of existence of five local regions in the asymptotic analysis.

Let us derive the governing fundamental differential equations corresponding to these regions from the vorticity equation (6). In region (I), time t is very small, so that we define the stretched time τ as $\tau = t/\varepsilon$. Then the governing equation, (6), becomes

$$\nabla^2\zeta - 2\frac{\partial\zeta}{\partial\tau} \approx 0.$$

In region (II), time t is of $O(1)$ and the space coordinates are of $O(1)$. Therefore, (6) becomes

$$\nabla^2\zeta \approx 0.$$

In region (III), time t is of $O(1)$ and the space coordinates are of $O(1/\varepsilon^{1/2})$, so that we set the stretched space coordinates as $(\hat{x}, \hat{y}) = \varepsilon^{1/2}(x, y)$. Then, we have

$$\nabla^2{}_{\hat{x}\hat{y}}\zeta - 2\frac{\partial \zeta}{\partial t} \approx 2f$$

where $\nabla^2{}_{\hat{x}\hat{y}} = \partial^2/\partial\hat{x}^2 + \partial^2/\partial\hat{y}^2$. In region (IV), time t is of $O(1/\varepsilon)$ and the space coordinates are of $O(1/\varepsilon)$, so that the stretched time and coordinates are defined as $\tau = \varepsilon t$ and $(X, Y) = \varepsilon(x, y)$. Then, we have

$$\nabla^2{}_{X,Y}\zeta - 2\frac{\partial \zeta}{\partial \tau} + \frac{\partial \zeta}{\partial X} \approx \frac{2}{\varepsilon}f.$$

In region (V), time t is of $O(1/\varepsilon)$ and the space coordinates are of $O(1)$, so that the stretched time τ is defined as $\tau = \varepsilon t$. Thus, we have

$$\nabla^2\zeta \approx 0.$$

The fundamental governing differential equation in region (II) is identical with that in region (V), that is, steady Stokes flow, and the fundamental equation in region (I) is also identical with that in region (III), that is, unsteady Stokes flow. However, the stretched coordinates are different for these two pairs of regions, so that the solutions in the two regions are different, as will be shown in the following sections. Thus, we see that five regions are necessary in the present problem and the application of the matched asymptotic expansion to integral equations is systematic. Further, we see that the generalized asymptotic expansions are easily derived, as shown in (27).

The present results show that (i) in the initial stage of motion the entire flow is essentially unsteady Stokes flow, (ii) with the development of time the unsteadiness develops downstream so that the flow near the cylinder is essentially steady Stokes but the far field is unsteady Stokes flow, so that Stokes–Oseen-like matching is necessary in this stage, and (iii) with further development of time the inertia term cannot be neglected far downstream so that the flow becomes of Stokes–Oseen type and the matching of Stokes–Oseen flow is necessary.

3.2. *Region (I)*

Region (I) is the flow in the case where p is very large, that is, the beginning of motion corresponding to the case that Bar-Lev & Yang (1975) analysed, using the method of matched asymptotic expansions with respect to the coordinate perturbation of the time axis. In this region we define a new variable $\hat{p}$ by

$$\hat{p} = (2\varepsilon p)^{1/2}. \tag{30}$$

Then, since $a = (\varepsilon^2 + \hat{p}^2)^{1/2}$ from (17), that is, $a = O(1)$, the degeneration of $\mathscr{K}_a$ is given by

$$\mathscr{K}_0\phi = \int_{r_s(\theta)}^{\infty} K_0(\hat{p}\rho_r)\phi(r_1)\mathrm{d}r_1. \tag{31}$$

The first approximation of F_0 is given by

$$F_0 \approx \frac{1}{2\pi}\oint_S \left[K_0(\hat{p}\rho_r)\frac{\partial\hat{\zeta}}{\partial n} - \hat{\zeta}\frac{\partial}{\partial n}K_0(\hat{p}\rho_r)\right]\mathrm{d}s.$$

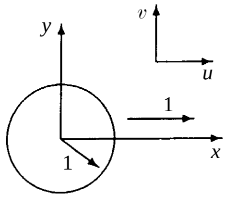

FIGURE 3. Low Reynolds number flow past a circular cylinder.

The vorticity $\hat{\zeta}$ is, therefore, given from (23) by

$$\hat{\zeta} = \frac{1}{2\pi} \oint_S \left[K_0(\hat{p}\rho_r) \frac{\partial \hat{\zeta}}{\partial n} - \hat{\zeta} \frac{\partial}{\partial n} K_0(\hat{p}\rho_r) \right] \mathrm{d}s + f_0^s + O(\varepsilon F). \quad (32)$$

From (24), the stream function $\tilde{\psi}$ is given by

$$\tilde{\psi} = \frac{1}{2\pi} \int_{r_s}^{\infty} \hat{\zeta} \log \left(\frac{1}{\rho_r} \right) r_1 \mathrm{d}r_1 + g_0^s + O(\varepsilon F). \quad (33)$$

Since $\tilde{\psi}$ is of $O(\varepsilon)$ from the boundary condition (20), $\tilde{\psi} = 2\varepsilon y/\hat{p}$ on S, $\hat{\zeta}$ and $\tilde{\psi}$ are of $O(\varepsilon)$, hence F is of $O(\varepsilon)$. Integrating (32) and (33) with respect to θ, we finally have

$$\hat{\zeta} = \frac{1}{2\pi} \oint_S \left[K_0(\hat{p}\rho_r) \frac{\partial \hat{\zeta}}{\partial n} - \hat{\zeta} \frac{\partial}{\partial n} K_0(\hat{p}\rho_r) \right] \mathrm{d}s + O(\varepsilon^2), \quad (34)$$

$$\tilde{\psi} = \frac{1}{2\pi} \int_0^{2\pi} \int_{r_s}^{\infty} \hat{\zeta} \log \left(\frac{1}{\rho_r} \right) r_1 \mathrm{d}r_1 \mathrm{d}\theta + O(\varepsilon^2). \quad (35)$$

The unknowns, $\partial\hat{\zeta}/\partial n$ and $\hat{\zeta}$ on S, are determined from the following boundary condition:

$$\tilde{\psi} = \frac{2\varepsilon y}{\hat{p}^2} \quad \text{on } r = r_s(\varphi), \quad (36)$$

$$\frac{\partial \tilde{\psi}}{\partial n} = \frac{2\varepsilon}{\hat{p}^2} \frac{\partial y}{\partial n} \quad \text{on } r = r_s(\varphi). \quad (37)$$

We apply the above results to the flow around an impulsively started circular cylinder, as shown in figure 3. The modified Bessel function is expressed as

$$K_0(\hat{p}\rho_r) = K_0(\hat{p}r) I_0(\hat{p}r_1) + 2 \sum_{m=1}^{\infty} K_m(\hat{p}r) I_m(\hat{p}r_1) \cos m(\theta - \varphi) \quad \text{for } r > r_1 \quad (38)$$

where I_m and K_m are the modified mth-order Bessel function of the first and second kind. Therefore, we have from (34) the first approximation of $\hat{\zeta}$:

$$\hat{\zeta} = \varepsilon \sum_{m=1}^{\infty} c_m K_m(\hat{p}r) \sin(m\varphi) \quad (39)$$

where c_m is given by

$$c_m = \frac{1}{\varepsilon} \frac{1}{\pi} \int_0^{2\pi} \left(-I_m(\hat{p}r_1) \frac{\partial \hat{\zeta}}{\partial r_1} + \hat{\zeta} \frac{\partial}{\partial r_1} I_m(\hat{p}r_1) \right)_{r_1=1} \sin(m\theta) \mathrm{d}\theta. \quad (40)$$

We note that c_m $(m = 1, 2, \ldots)$ is at most of $O(1)$, since $\hat{\zeta}$ is of $O(\varepsilon)$.

We substitute (39) into (35), then the stream function is obtained as follows:

$$\tilde{\psi} \approx \varepsilon \frac{1}{2} \sum_{m=1}^{\infty} c_m \frac{1}{m} \left[\frac{1}{\hat{p}} \left\{ -\frac{2m}{\hat{p}} K_m(\hat{p}r) + \left(\frac{1}{r}\right)^m K_{m+1}(\hat{p}) \right\} \right] \sin(m\varphi). \tag{41}$$

We here used the following relation:

$$-\log \frac{1}{\rho_r} = \begin{cases} \log r - \displaystyle\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_1}{r}\right)^m \cos m(\theta - \varphi) & \text{for } \dfrac{r_1}{r} < 1 \\ \log r_1 - \displaystyle\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r}{r_1}\right)^m \cos m(\theta - \varphi) & \text{for } \dfrac{r}{r_1} < 1. \end{cases} \tag{42}$$

The constants c_m, therefore, are determined from the boundary condition (36):

$$c_1 = \frac{4}{\hat{p} K_0(\hat{p})}, \qquad c_m = 0 \quad (m \geqslant 2).$$

Thus, $\tilde{\psi}$ and $\hat{\zeta}$ in this region are finally obtained as

$$\tilde{\psi} \approx \frac{2\varepsilon}{\hat{p}^2} \frac{1}{K_0(\hat{p})} \left\{ -\frac{2}{\hat{p}} K_1(\hat{p}r) + \frac{K_2(\hat{p})}{r} \right\} \sin \varphi, \tag{43}$$

$$\hat{\zeta} \approx \frac{4\varepsilon}{\hat{p}} \frac{K_1(\hat{p}r)}{K_0(\hat{p})} \sin \varphi. \tag{44}$$

We note that the boundary condition (37) is automatically satisfied. Details of the analysis are in the typescript in the editorial files.

3.3. *Regions (II) and (III)*

With the increase of time the viscous force begins to influence the far-field flow. At this time the fundamental equation of the flow in region (III) is unsteady Stokes equation; however, matching of the Stokes–Oseen type is necessary to get a uniform approximate solution.

In regions (II) and (III), p is of $O(1)$, so that we define p_0 as

$$p_0 \equiv 2p. \tag{45}$$

Then, since $a = \{\varepsilon(\varepsilon + p_0)\}^{1/2}$ from (17), we easily see that the significant local variable r^* of the integral operator $\mathscr{K}_a$ is given by

$$r^* = \varepsilon^{1/2} r. \tag{46}$$

The degeneration of $\mathscr{K}_a$ in region (II) is given from (27) by

$$\mathscr{K}_0 \phi = -\int_{r_s(\theta)}^{\infty} \left(\gamma + \log \frac{(\varepsilon p_0)^{1/2}}{2} + \log \rho_r \right) \phi(r_1) \mathrm{d}r_1. \tag{47}$$

In this case, the function F_0 becomes

$$F_0 = \frac{1}{2\pi} \oint_S \left\{ \frac{\partial \hat{\zeta}}{\partial n} \left(-\gamma - \log \frac{(\varepsilon p_0)^{1/2}}{2} + \log \frac{1}{\rho_r} \right) - \hat{\zeta} \frac{\partial}{\partial n} \log \frac{1}{\rho_r} \right\} \mathrm{d}s + O(\varepsilon).$$

Kida & Miyai (1973) proposed a method to obtain an asymptotic expansion of an integral equation whose kernel has a small parameter, in the case of the singular perturbation problem. This concept is proved to be reasonable by Kida (1991). They assume that the integrand in the outer region is indeterminate when the asymptotic expansion is obtained on the inner region. By using this concept, let us try to obtain the asymptotic expansion of the following integral I_f with fixed r:

$$I_f = \int_0^{2\pi} \int_{r_s(\theta)}^{\infty} F(r_1, \theta) K_0((\varepsilon p_0)^{1/2} \rho_r) r_1 \mathrm{d}r_1 \mathrm{d}\theta. \tag{48}$$

We divide the integral domain into two parts:

$$I_f = \int_0^{2\pi} \int_{r_s(\theta)}^{1/\delta_o} F K_0\left((\varepsilon p_0)^{1/2} \rho_r\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta + \int_0^{2\pi} \int_{1/\delta_o}^{\infty} F K_0\left((\varepsilon p_0)^{1/2} \rho_r\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta$$

where $\delta_o = \varepsilon^{1/2+s}$ for some $s > 0$. When we obtain the asymptotic expansion in region (II), Kida (1991) shows that it is reasonable to estimate the above integral under the assumption that F for $1/\delta_o \leqslant r_1$ is indeterminate. Therefore, we have

$$I_f = \int_0^{2\pi} \int_{r_s(\theta)}^{1/\delta_o} F K_0\left((\varepsilon p_0)^{1/2} \rho_r\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta + \int_0^{2\pi} \int_{1/\delta_o}^{\infty} F \Bigg[K_0\left((\varepsilon p_0)^{1/2} r_1\right) I_0\left((\varepsilon p_0)^{1/2} r\right)$$
$$+ 2 \sum_{m=1}^{\infty} K_m\left((\varepsilon p_0)^{1/2} r_1\right) I_m\left((\varepsilon p_0)^{1/2} r\right) \cos m(\theta - \varphi) \Bigg] r_1 \mathrm{d}r_1 \mathrm{d}\theta.$$

Taking into account that F is odd function of θ, we have from the above equation:

$$I_f = \int_0^{2\pi} \int_{r_s(\theta)}^{1/\delta_o} F K_0\left((\varepsilon p_0)^{1/2} \rho_r\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta + \sum_{m=1}^{\infty} \tilde{C}_m^s r^m \sin(m\varphi)$$

where

$$\begin{aligned} \tilde{C}_m^s &= O\left[(\varepsilon p_0)^{m/2} \int_0^{2\pi} \int_{1/\delta_o}^{\infty} K_m\left((\varepsilon p_0)^{1/2} r_1\right) F \sin(m\theta) r_1 \mathrm{d}r_1 \mathrm{d}\theta \right] \\ &= O\left(\varepsilon^{m/2} \int_{1/\delta_o}^{\infty} \left(\frac{1}{\varepsilon^{1/2} r_1} \right)^m r_1 \mathrm{d}r_1 F \right) = O(\delta_o^{m-2} F) \\ &\approx O(\varepsilon^{m/2-1} F). \end{aligned}$$

Thus, $\tilde{C}_m^s$ is an indeterminate constant being of $O(\varepsilon^{m/2-1}F)$, because F is indeterminate for $1/\delta_o \leqslant r_1$. The first term of the right-hand side of the above equation for I_f is rewritten as

$$\begin{aligned} (\text{first term}) \approx & \int_0^{2\pi} \mathrm{Pf} \int_{r_s(\theta)}^{\infty} F \left(-\gamma - \log \frac{(\varepsilon p_0)^{1/2}}{2} + \log \frac{1}{\rho_r} \right) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\ & - \int_0^{2\pi} \mathrm{Pf} \int_{1/\delta_o}^{\infty} F \Bigg(-\gamma - \log \frac{(\varepsilon p_0)^{1/2}}{2} - \log r_1 \\ & + \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r}{r_1} \right)^m \cos m(\theta - \varphi) \Bigg) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\ \approx & \int_0^{2\pi} \mathrm{Pf} \int_{r_s(\theta)}^{\infty} F \left(-\gamma - \log \frac{(\varepsilon p_0)^{1/2}}{2} + \log \frac{1}{\rho_r} \right) r_1 \mathrm{d}r_1 \mathrm{d}\theta + \sum_{m=1}^{\infty} \hat{C}_m^s r^m \sin(m\varphi) \end{aligned}$$

where $\hat{C}_m^s$ is a constant of $O(\varepsilon^{m/2-1}F)$. Integral sign, $\mathrm{Pf}\int_{r_s}^{\infty}(\ )\mathrm{d}r_1$ denotes the finite part of $\lim_{\delta_0\to 0}\int_{r_s}^{1/\delta_0}(\ )\mathrm{d}r_1$. Therefore, we finally arrive at

$$\hat{\zeta} \approx \frac{1}{2\pi}\oint_s \left\{\frac{\partial\hat{\zeta}}{\partial n}\log\frac{1}{\rho_r} - \hat{\zeta}\frac{\partial}{\partial n}\log\frac{1}{\rho_r}\right\}\mathrm{d}s + \frac{\varepsilon}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{\infty} F\log(\rho_r)\, r_1\mathrm{d}r_1\mathrm{d}\theta - \frac{\varepsilon^{1/2}}{2}\sum_{m=1}^{\infty} C_m^s r^m \sin(m\varphi) \tag{49}$$

where $C_m^s = \varepsilon^{1/2}(\tilde{C}_m^s + \hat{C}_m^s)/\pi$. Since $\tilde{C}_m^s$ is indeterminate and of $O(\varepsilon^{m/2-1}F)$, therefore C_m^s is at this stage an indeterminate constant of $O(\varepsilon^{m/2-1/2})$, which is determined by the matching process. Following similar steps, we arrive at

$$\tilde{\psi} \approx \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{\infty}\hat{\zeta}\log\left(\frac{1}{\rho_r}\right)r_1\mathrm{d}r_1\mathrm{d}\theta - \frac{\varepsilon}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{\infty}\hat{\zeta}\log\left(\frac{1}{\rho_r}\right)r_1^2\cos(\theta)r_1\mathrm{d}r_1\mathrm{d}\theta - \frac{1}{2}\sum_{m=1}^{\infty}E_m^s r^m\sin(m\varphi) \tag{50}$$

where E_m^s is also an indeterminate constant of $O(\varepsilon^{m/2-1}\hat{\zeta})$ which is determined by the matching process.

In region (III), the significant local operator $\mathscr{K}_a^*$ of $\mathscr{K}_a$ is given by

$$\mathscr{K}_a^*\phi^* = \int_{\varepsilon^{1/2}r_s(\theta)}^{\infty} K_0\left((\varepsilon+p_0)^{1/2}\rho_r^*\right)\phi^*(r_1^*)\mathrm{d}r_1^*$$

where $\rho_r^* = (r^{*2} + r_1^{*2} - 2r^*r_1^*\cos(\theta-\varphi))^{1/2}$. The degeneration of $\mathscr{K}_a^*$ is given by

$$\mathscr{K}_0^*\phi^* = \int_0^{\infty} K_0\left(p_0^{1/2}\rho_r^*\right)\phi^*\mathrm{d}r_1^*. \tag{51}$$

Using relation (38), we have the function F_0 from (22) and the symmetry of flow as follows:

$$F_0 = (\varepsilon p_0)^{1/2}C_0K_1\left(p_0^{1/2}r^*\right)\sin\varphi + O(\varepsilon) \tag{52}$$

where

$$C_0 \approx \frac{1}{2\pi}\oint_S\left\{\frac{\partial\hat{\zeta}}{\partial n}r_s\sin\theta_s - \hat{\zeta}\frac{\partial}{\partial n}(r_s\sin\theta_s)\right\}\mathrm{d}s.$$

Let us consider the integral I_f defined by (48) for the significant local variable r^*. In this variable, I_f can be expressed as

$$I_f = \frac{1}{\varepsilon}\int_0^{2\pi}\left(\int_{\delta_o}^{\infty} + \int_{\varepsilon^{1/2}r_s}^{\delta_o}\right)FK_0\left(p_0^{1/2}\rho_r^*\right)r_1^*\mathrm{d}r_1^*\mathrm{d}\theta$$

where δ_o is of order of $\varepsilon^{1/2-s}$ for some s ($s > 0$). In this region, F for $\varepsilon^{1/2}r_s \leqslant r^* \leqslant \delta_o$ is indeterminate, so that we follow similar steps as in region (II) and we finally arrive at, by using (38),

$$\hat{\zeta}^* \approx C_0 p_0^{1/2} K_1\left(p_0^{1/2} r^*\right)\sin\varphi - \frac{\varepsilon^{1/2}}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_0^{\infty} F^* K_0\left(p_0^{1/2}\rho_r^*\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta$$

$$+\frac{\varepsilon^{1/2}}{2}\sum_{m=1}^{\infty} D_m K_m\left(p_0^{1/2} r^*\right)\sin(m\varphi), \tag{53}$$

$$\tilde{\psi}^* \approx \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_0^{\infty} \hat{\zeta}^* \log\left(\frac{1}{\rho_r^*}\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta$$

$$-\frac{\varepsilon^{1/2}}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_0^{\infty} \hat{\zeta}^* \log\left(\frac{1}{\rho_r^*}\right) r_1^{*2}\cos\theta \mathrm{d}r_1^* \mathrm{d}\theta + \frac{1}{2}\sum_{m=1}^{\infty}\frac{H_m}{r^{*m}}\sin(m\varphi) \tag{54}$$

where D_m and H_m are at this stage indeterminate constants of $O(\varepsilon^{m/2+1}F^*)$ and $O(\varepsilon^{m/2+1}\hat{\zeta}^*)$ respectively, which are determined by the matching process. Integral sign, $\mathrm{Pf}\int_0^{\infty}(\ )\mathrm{d}r_1^*$, denotes the finite part of $\lim_{\delta_0\to 0}\int_{\delta_0}^{\infty}(\ )\mathrm{d}r_1^*$. We see from (53) and (54) that the fundamental governing equation of this flow is of Stokes type, because in (54) the $\exp(-r_1^*\cos\theta)$ term is not in the integrand and its effect appears in the second term of the right-hand side of this equation. Here $\hat{\zeta}^*$, $\tilde{\psi}^*$, and F^* are defined by

$$\hat{\zeta}^* = \frac{\hat{\zeta}}{\varepsilon^{1/2}},\quad \tilde{\psi}^* = \varepsilon^{1/2}\tilde{\psi},\quad F^* = \frac{F}{\varepsilon}. \tag{55}$$

Details of the derivation of (53) and (54) are in the typescript in the editorial files.

Let us consider the flow around the circular cylinder. From (49) and (50), the solutions in region (II) are given by

$$\hat{\zeta} \approx \sum_{m=1}^{\infty} a_m \sin(m\varphi)\left(\frac{1}{r}\right)^m, \tag{56}$$

$$\tilde{\psi} \approx \frac{1}{2}a_1\sin\varphi\left(\frac{1}{r}\frac{r^2-1}{2} - r\log r\right)$$

$$+\frac{1}{2}\sum_{m=2}^{\infty} a_m \frac{\sin(m\varphi)}{m}\left[\left(\frac{1}{r}\right)^m \frac{r^2-1}{2} + \frac{r^m}{2m-2}\frac{r^2}{r^{2m}}\right] - \frac{1}{2}\sum_{m=1}^{\infty} E_m^s r^m \sin(m\varphi) \tag{57}$$

where

$$a_m = -\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{m}\frac{\partial\hat{\zeta}}{\partial r} - \hat{\zeta}\right)_{r=1}\sin(m\theta)\mathrm{d}\theta.$$

Since the boundary condition (20) of $\tilde{\psi}$ becomes $\tilde{\psi} = (2/p_0)\sin\varphi$ for $r = 1$, we have

$$a_m \approx 0 \quad (m \neq 1), \tag{58}$$

$$E_1^s \approx -\frac{4}{p_0} \ \text{ and } \ E_m^s \approx 0 \quad (m \neq 1). \tag{59}$$

Thus, we have

$$\hat{\zeta} \approx \frac{a_1}{r}\sin\varphi, \tag{60}$$

$$\tilde{\psi} \approx \left[\frac{1}{2}a_1\left\{\frac{1}{r}\frac{r^2-1}{2} - r\log r\right\} + \frac{1}{p}r\right]\sin\varphi. \tag{61}$$

Let us consider the matching between region (II) and region (III). As $r \to \infty$, $\hat{\zeta} \to (a_1/r)\sin\varphi$ and $\hat{\zeta}^* \to (C_0/r^*)\sin\varphi$ for $r^* \to 0$ from (53), we have the following relation from the matching requirement of $\hat{\zeta}$ between region (II) and region (III):

$$C_0 = a_1, \quad D_m \approx 0.$$

The stream function $\tilde{\psi}$ as $r \to \infty$ is given from (61) by

$$\tilde{\psi} \to \left[\frac{1}{2}a_1\left\{\frac{r}{2} - r\log r\right\} + \frac{r}{p}\right]\sin\varphi.$$

On the other hand, the stream function $\tilde{\psi}^*$ is obtained by substituting (53) into (54):

$$\tilde{\psi}^* \approx \frac{C_0}{2\pi}p_0^{1/2}\int_0^{\infty} \mathrm{Pf}\int_0^{\infty} K_1\left(p_0^{1/2}r_1^*\right)\sin\theta\log\left(\frac{1}{\rho_r^*}\right)r_1^*\mathrm{d}r_1^*\mathrm{d}\theta + \frac{H_1}{2r^*}\sin\varphi.$$

Therefore, we have

$$\begin{aligned}\tilde{\psi}^* &\approx \frac{C_0}{2}p_0^{1/2}\left[\mathrm{Pf}\int_0^{r^*} K_1\left(p_0^{1/2}r_1^*\right)\frac{r_1^*}{r^*}r_1^*\mathrm{d}r_1^* + r^*\,\mathrm{Pf}\int_{r^*}^{\infty} K_1\left(p_0^{1/2}r_1^*\right)r_1^*\mathrm{d}r_1^*\right]\sin\varphi + \frac{H_1}{2r^*}\sin\varphi \\ &\approx \frac{C_0}{2}p_0^{1/2}\left[\frac{r^*}{p_0^{1/2}}K_0\left(p_0^{1/2}r^*\right) - \frac{r^*}{p_0^{1/2}}K_2\left(p_0^{1/2}r^*\right) + \frac{2}{p_0^{3/2}}\frac{1}{r^*}\right]\sin\varphi + \frac{H_1}{2r^*}\sin\varphi. \qquad (62)\end{aligned}$$

From (62), we have as $r^* \to 0$

$$\tilde{\psi}^* \approx \frac{C_0}{2}p_0^{1/2}\left[\frac{r^*}{p_0^{1/2}}\left(-\gamma - \log\frac{p_0^{1/2}}{2} - \log r^*\right) + \frac{r^*}{2p_0^{1/2}}\right]\sin\varphi + \frac{H_1}{2r^*}\sin\varphi.$$

Thus we can determine the constants C_0 and H_1 by the matching requirement of the stream function:

$$C_0 \approx -\frac{4}{p}\frac{1}{2\left(\gamma + \frac{1}{2}\log\varepsilon\right) + \log p - \log 2}, \qquad (63)$$

$$H_1 \approx -\tfrac{1}{2}\varepsilon C_0. \qquad (64)$$

Details of derivation of (56)–(64) are in the typescript in the editorial files. From these results, we finally obtain the following relations:

Case (II):

$$\hat{\zeta} \approx \frac{C_0}{r}\sin\varphi, \qquad (65)$$

$$\tilde{\psi} \approx \frac{C_0}{2}\left(\frac{r^2-1}{2r} - r\log r\right)\sin\varphi + \frac{r}{p}\sin\varphi; \qquad (66)$$

Case (III):

$$\hat{\zeta} \approx C_0(2p\varepsilon)^{1/2}K_1\left((2p)^{1/2}r^*\right)\sin\varphi, \qquad (67)$$

$$\tilde{\psi} \approx \frac{C_0}{2\varepsilon^{1/2}}\left\{r^*K_0\left((2p)^{1/2}r^*\right) - r^*K_2\left((2p)^{1/2}r^*\right) + \frac{1}{pr^*}\right\}\sin\varphi - \frac{\varepsilon^{1/2}}{4}\frac{C_0}{r^*}\sin\varphi. \quad (68)$$

3.4. *Regions (IV) and (V)*

As time further increases, the viscous force influences the far flow field and the region where the inertia force is comparable with the viscous force extends to the far field.

For this case, p is of $O(1/\varepsilon)$. We define $\tilde{p}$ by

$$\tilde{p} = \frac{2p}{\varepsilon}. \tag{69}$$

The integral operator $\mathscr{K}_a$ becomes

$$\mathscr{K}_a\phi = \int_{r_s(\theta)}^{\infty} K_0\left(\varepsilon(1+\tilde{p})^{1/2}\rho_r\right)\phi(r_1)\mathrm{d}r_1. \tag{70}$$

Therefore, the significant variable R is given by

$$R = \varepsilon r. \tag{71}$$

Then, the function F_0 in region (IV) becomes from (22)

$$F_0 \approx \varepsilon(1+\tilde{p})^{1/2}\,\hat{C}_0 K_1\left((1+\tilde{p})^{1/2}R\right)\sin\varphi \tag{72}$$

where

$$\hat{C}_0 \approx \frac{1}{2\pi}\oint_S\left[\frac{\partial\hat{\zeta}}{\partial n}r_s\sin\theta_s - \hat{\zeta}\frac{\partial}{\partial n}(r_s\sin\theta_s)\right]\mathrm{d}s.$$

We note that $\hat{C}_0$ is the first approximation of the right-hand side of the above equation with fixed $\tilde{p}$ but C_0 given by (52) is one with fixed p_0.

Following the procedure mentioned in §3.3, we finally arrive at the following relations in this region:

$$\hat{\zeta}^* \approx \alpha_1 K_1\left((1+\tilde{p})^{1/2}R\right)\sin\varphi - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}F^* K_0\left((1+\tilde{p})^{1/2}\rho_R^*\right)R_1\mathrm{d}R_1\mathrm{d}\theta$$
$$+\sum_{m=0}^{\infty}E_m K_m\left((1+\tilde{p})^{1/2}R\right)\sin(m\varphi), \tag{73}$$

$$\tilde{\psi}^* \approx \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}\exp\left[-R_1\cos\theta\right]\hat{\zeta}^*\log\left(\frac{1}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta \tag{74}$$

where

$$\alpha_1 = (1+\tilde{p})^{1/2}\hat{C}_0,$$
$$\rho_R^* = (R^2+R_1^2-2RR_1\cos\theta)^{1/2}.$$

Integral sign, $\mathrm{Pf}\int_0^{\infty}(\quad)\mathrm{d}R_1$ denotes the finite part of $\lim_{\delta_0\to 0}\int_{\delta_0}^{\infty}(\quad)\mathrm{d}R_1$.

In (74), the term $\exp(-R_1\cos\theta)$ is in the integrand, so that the essential equation in this region is of Oseen type. Here, $\hat{\zeta}^*$ and $\tilde{\psi}^*$ are defined by

$$\hat{\zeta}^* = \frac{\hat{\zeta}}{\varepsilon},\quad \tilde{\psi}^* = \varepsilon\tilde{\psi},\quad F^* = \frac{F}{\varepsilon^2}.$$

Furthermore, we obtain the solutions in region (V):

$$\hat{\zeta} \approx \frac{1}{2\pi}\oint_S\left[\sigma_0\frac{\partial}{\partial n}\log\rho_r - \mu_0\log\rho_r\right]\mathrm{d}s, \tag{75}$$

$$\tilde{\psi} \approx -r\sin\varphi\left[\frac{\alpha_1}{2(1+\tilde{p})^{1/2}}\log\varepsilon - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}\hat{\zeta}^*\exp(-R_1\cos\theta)\sin\theta\mathrm{d}R_1\mathrm{d}\theta\right]$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{\infty}\hat{\zeta}\log\left(\frac{1}{\rho_r}\right)r_1\mathrm{d}r_1\mathrm{d}\theta \qquad (76)$$

where $\sigma_0 = \hat{\zeta}$, and $\mu_0 = \partial\hat{\zeta}/\partial n$ on S. We note that the second term in square brackets in the right hand side of (76) is derived from the matching procedure between the solutions in regions (IV) and (V) (see Kida & Take 1992*a*).

For the circular cylinder, the solutions in region (V) are given from (75) and (76) by

$$\hat{\zeta} \approx \sum_{m=1}^{\infty}\frac{\tilde{C}_m}{r^m}\sin(m\varphi), \qquad (77)$$

$$\tilde{\psi} \approx -r\sin\varphi\left[\frac{\alpha_1}{2(1+\tilde{p})^{1/2}}\log\varepsilon - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}\hat{\zeta}_0^*\exp(-R_1\cos\theta)\sin\theta\mathrm{d}R_1\mathrm{d}\theta\right]$$
$$+\frac{1}{2}\tilde{C}_1\sin\varphi\left[\frac{r^2-1}{2r} - r\log r\right] + \frac{1}{2}\sum_{m=2}^{\infty}\frac{\tilde{C}_m}{m}\sin(m\varphi)\left[\frac{r^2-1}{2r^m} + \frac{r^2}{2m-2}\frac{1}{r^m}\right] \qquad (78)$$

where $\tilde{C}_m$ is constant. From the boundary condition (20), we easily see that

$$\tilde{C}_m = 0 : \quad m \geqslant 2, \qquad (79)$$

$$\alpha_1 \approx -\frac{1}{p}\Bigg/\left(\frac{1}{2(1+\tilde{p})^{1/2}}\log\varepsilon - \mathrm{Pf}\int_0^{\infty}\frac{K_1\left((1+\tilde{p})^{1/2}R_1\right)I_1(R_1)}{R_1}\mathrm{d}R_1\right). \qquad (80)$$

Assuming that $F^* \sim o(\alpha_1)$, we can get $\tilde{C}_1$ from the matching requirement between the solutions in regions (IV) and (V) with respect to $\hat{\zeta}$:

$$\tilde{C}_1 = \frac{\alpha_1}{(1+\tilde{p})^{1/2}}.$$

From these results, $\alpha_1 \sim O(1/\log\varepsilon)$, so that $F^* \sim O(1/\log^2\varepsilon)$. Thus, we see that the above assumption is reasonable.

Summarizing these results, we finally arrive at the following relations:
for case (IV)

$$\hat{\zeta} \approx \varepsilon\alpha_1 K_1\left((1+\tilde{p})^{1/2}R\right)\sin\varphi, \qquad (81)$$

$$\tilde{\psi} \approx \frac{1}{2\pi\varepsilon}\int_0^{2\pi}\int_0^{\infty}\exp(-R_1\cos\theta)\frac{\hat{\zeta}}{\varepsilon}\log\left(\frac{1}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta; \qquad (82)$$

for case (V)

$$\hat{\zeta} \approx \frac{\tilde{C}_1}{r}\sin\varphi, \qquad (83)$$

$$\tilde{\psi} \approx \frac{r}{p}\sin\varphi + \frac{\tilde{C}_1}{2}\left(\frac{r^2-1}{2r} - r\log r\right)\sin\varphi. \qquad (84)$$

We note that α_1 is given by

$$\alpha_1 = -\frac{(1+\tilde{p})^{1/2}}{p}\frac{2}{\gamma + \log\left(\frac{1}{2}\varepsilon\right) + A}, \tag{85}$$

$$A = \frac{1}{2}\log(1+\tilde{p}) - \frac{1}{2}\left(-\tilde{p}\log\left(\frac{1+\tilde{p}}{\tilde{p}}\right) + 1\right). \tag{86}$$

Further, we note that $\tilde{C}_1$ is not the same as C_0 in case (II), that is, the Stokes solution in region (II) is different from that in region (V). Details of the analysis in this section are in the typescript in the editorial files.

4. Composite solution

In this section, we consider the composite solutions of the flow around a circular cylinder with respect to the Laplace variable p, to obtain the stream function in the real variables. As we see from figure 2, we must obtain the composite solutions with respect to p for the three space variables r, r^*, and R.

We here denote the stream functions expressed by the Laplace variable p in regions (I), (II), (III), (IV) and (V) as $\tilde{\psi}_I$, $\tilde{\psi}_{II}$, $\tilde{\psi}_{III}$, $\tilde{\psi}_{IV}$ and $\tilde{\psi}_V$, respectively. Further, we denote the asymptotic expansion of $\tilde{\psi}_I$ with respect to p as $\tilde{\psi}_{I\to II}$. Similarly, we can define the asymptotic expansion of $\tilde{\psi}_{II}$ with respect to $\hat{p}$ as $\tilde{\psi}_{II\to I}$. The requirement of the matching is $\tilde{\psi}_{I\to II} = \tilde{\psi}_{II\to I}$. We can confirm the requirement of the matching with respect to p for each variable r, r^* and R.

For the variable r, we have $\tilde{\psi}_{I\to II} = \tilde{\psi}_{II}$ and $\tilde{\psi}_{V\to II} = \tilde{\psi}_{II}$. Therefore, we can arrive at

$$\begin{aligned}
\tilde{\psi}^c &= \tilde{\psi}_I + \tilde{\psi}_V - \tilde{\psi}_{II} \\
&= \frac{1}{p}\frac{1}{K_0\left((2\varepsilon p)^{1/2}\right)}\left(-\frac{2}{(2\varepsilon p)^{1/2}}K_1\left((2\varepsilon p)^{1/2}r\right) + \frac{1}{r}K_2\left((2\varepsilon p)^{1/2}\right)\right)\sin\varphi \\
&\quad + \left(\frac{\tilde{C}_1}{2} - \frac{C_0}{2}\right)\left(\frac{r}{2} - \frac{1}{2r} - r\log r\right)\sin\varphi.
\end{aligned}$$

For the variable r^*, we have

$$\begin{aligned}
\tilde{\psi}^c &= \tilde{\psi}_I + \tilde{\psi}_{III} + \tilde{\psi}_V - \tilde{\psi}_{I\to III} - \tilde{\psi}_{III\to V} \\
&= \frac{1}{p}\frac{K_2\left((2\varepsilon p)^{1/2}\right)}{K_0\left((2\varepsilon p)^{1/2}\right)}\frac{\varepsilon^{1/2}}{r^*}\sin\varphi - \frac{C_0}{(2\varepsilon p)^{1/2}}K_1\left((2p)^{1/2}r^*\right)\sin\varphi \\
&\quad + \left(\frac{\tilde{C}_1}{2} - \frac{C_0}{2}\right)\left(\frac{r^*}{2\varepsilon^{1/2}} - \frac{\varepsilon^{1/2}}{2r^*} - \frac{r^*}{\varepsilon^{1/2}}\log\frac{r^*}{\varepsilon^{1/2}}\right)\sin\varphi.
\end{aligned}$$

For the variable R, we have

$$\begin{aligned}
\tilde{\psi}^c &= \tilde{\psi}_I + \tilde{\psi}_{III} + \tilde{\psi}_{IV} - \tilde{\psi}_{I\to III} - \tilde{\psi}_{III\to IV} \\
&= \frac{1}{p}\frac{\varepsilon}{R}\frac{K_2\left((2\varepsilon p)^{1/2}\right)}{K_0\left((2\varepsilon p)^{1/2}\right)}\sin\varphi - \frac{C_0}{2p}\frac{1}{R}\sin\varphi \\
&\quad + \frac{\alpha_1}{2\pi\varepsilon}\int_0^{2\pi}\int_0^{\infty}\exp(-R_1\cos\theta) \\
&\quad \times K_1\left((1+\tilde{p})^{1/2}R_1\right)\sin\theta\log\left(\frac{1}{\rho^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta.
\end{aligned}$$

From these relations, we can get the stream function in real space using the following relation:

$$\psi = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \exp(pt)\tilde{\psi}\mathrm{d}p \quad c > 0. \tag{87}$$

It is however very difficult to obtain the analytical expressions for the stream function from the above relation, so that in the present paper a numerical calculation is carried out. In order to do this, we have to change the above contour of integration to a contour which consists of the part of real axis from $-\infty - \mathrm{i}0$ to $-\infty + \mathrm{i}0$ passing through a small circle having its centre at the origin and two small semicircles around the origin $p = -\varepsilon/2$. We further use the following relations of the modified Bessel functions (Erdelyi *et al.* 1953):

$$K_0\left(x \exp\left(\pm\tfrac{1}{2}\pi \mathrm{i}\right)\right) = \mp\tfrac{1}{2}\mathrm{i}\pi\left(J_0(x) \mp \mathrm{i}Y_0(x)\right),$$
$$K_1\left(x \exp\left(\pm\tfrac{1}{2}\pi \mathrm{i}\right)\right) = -\tfrac{1}{2}\pi\left(J_1(x) \mp \mathrm{i}Y_1(x)\right).$$

Finally, we have the following relation for r:

$$\begin{aligned}\psi \approx{}& rH(t)\sin\varphi - \frac{1}{\pi}\left(\frac{2}{\varepsilon}\right)^{1/2} \int_0^\infty \frac{\exp(-xt)}{x^{3/2}} \frac{1}{J_0^2(X) + Y_0^2(X)} \\ &\times \left(\frac{1}{r}\left(J_1(X)Y_0(X) - J_0(X)Y_1(X)\right) - \left(Y_0(X)J_1(Xr) - J_0(X)Y_1(Xr)\right)\right)\mathrm{d}x \sin\varphi \\ &+ \left(\frac{r}{2} - \frac{1}{2r} - r\log r\right) F_2 \sin\varphi \end{aligned} \tag{88}$$

where J_i and Y_i are Bessel functions of the first and second kind, $X = (2\varepsilon x)^{1/2}$, and F_2 is defined as follows:

$$F_2 = F_{21} - F_{22}$$

where F_{21} and F_{22} are given by

$$\begin{aligned} F_{21} ={}& -\frac{1}{2}\int_0^\infty \frac{\exp(-xt)}{x} \frac{\mathrm{d}x}{\left(\gamma + \log\frac{1}{2}\varepsilon^{1/2}(2x)^{1/2}\right)^2 + \frac{1}{4}\pi^2}, \\ F_{22} ={}& \frac{H(t)}{\gamma + \log\frac{1}{2}\varepsilon - \frac{1}{2}} - \frac{1}{2}\int_{\varepsilon/2}^\infty \frac{\exp(-xt)}{x} \\ &\times \frac{\mathrm{d}x}{\left(\gamma + \frac{1}{2}\log\frac{1}{2}\varepsilon + \frac{1}{2}\log\left(x - \frac{1}{2}\varepsilon\right) - \frac{1}{2} + \dfrac{x}{\varepsilon}\log\left(\dfrac{x}{x - \frac{1}{2}\varepsilon}\right)\right)^2 + \dfrac{1}{4}\pi^2} \\ &-\frac{1}{\varepsilon}\int_0^{\varepsilon/2} \exp(-xt) \\ &\times \frac{\mathrm{d}x}{\left(\gamma + \frac{1}{2}\log\frac{1}{2}\varepsilon + \frac{1}{2}\log\left(\frac{1}{2}\varepsilon - x\right) - \frac{1}{2} + \dfrac{x}{\varepsilon}\log\left(\dfrac{x}{\frac{1}{2}\varepsilon - x}\right)\right)^2 + \dfrac{\pi^2 x^2}{\varepsilon^2}}. \end{aligned}$$

For r^*, we have

$$\psi \approx \frac{\varepsilon^{-1/2}}{r^*}\frac{1}{2\pi}\int_0^\infty \frac{\exp(-xt)}{x^2}\left[\frac{\pi}{\frac{1}{4}\pi^2+\left(\gamma+\frac{1}{2}\log\frac{1}{2}\varepsilon x\right)^2}\right.$$
$$\left.-\frac{4}{\pi}\frac{1}{J_0(X_1)^2+Y_0(X_1)^2}\right]\mathrm{d}x\sin\varphi+\left[\frac{r^*}{\varepsilon^{1/2}}+\int_0^\infty\frac{\exp(-xt)}{xX_1}\right.$$
$$\left.\times\left(\frac{J_1(X_2r^*)\left(\gamma+\frac{1}{2}\log\frac{1}{2}\varepsilon x\right)-\frac{1}{2}\pi Y_1(X_2r^*)-\frac{1}{(2x)^{1/2}}\frac{1}{r^*}}{\left(\gamma+\frac{1}{2}\log\frac{1}{2}\varepsilon x\right)^2+\frac{1}{4}\pi^2}\right)\mathrm{d}x\right]\sin\varphi$$
$$+(F_{21}-F_{22})\left(\frac{r^*}{\varepsilon^{1/2}}-\frac{\varepsilon^{1/2}}{2r^*}-\frac{r^*}{\varepsilon^{1/2}}\log\frac{r^*}{\varepsilon^{1/2}}\right)\sin\varphi \tag{89}$$

where $X_1=(2\varepsilon x)^{1/2}$ and $X_2=(2x)^{1/2}$.
For R, we have

$$\psi \approx \frac{1}{R}\frac{1}{2\pi}\int_0^\infty\frac{\exp(-xt)}{x^2}\left[\frac{\pi}{\frac{1}{4}\pi^2+\left(\gamma+\frac{1}{2}\log\frac{1}{2}\varepsilon x\right)^2}-\frac{4}{\pi}\frac{1}{J_0(X_1)^2+Y_0(X_1)^2}\right]\mathrm{d}x\sin\varphi$$
$$+\frac{1}{\varepsilon^2}\sum_{m=1}^{\infty}\sin(m\varphi)\left[\int_0^t\frac{1}{s^2}\exp(-\tfrac{1}{2}\varepsilon s)F_{22}(t-s)\mathrm{d}s\left(\frac{1}{R^m}\int_0^R\exp\left(-\frac{R_1^2}{2\varepsilon s}\right)R_1^{m+1}I_m(-R_1)\mathrm{d}R_1\right.\right.$$
$$\left.\left.+R^m\int_R^\infty\exp\left(-\frac{R_1^2}{2\varepsilon s}\right)R_1^{-m+1}I_m(-R_1)\mathrm{d}R_1\right)\right]. \tag{90}$$

There are integrals in the above relations which are singular, so that in the actual numerical calculation we have used the asymptotic expressions near $x=0$, that is, we subtract the asymptotic expressions from the integrand and the integration of the asymptotic terms is carried out changing the integral variable. The details of the formation of the composite solutions are given in the typescript in the editorial files.

5. Aerodynamic force

The aerodynamic force on the cylinder is calculated from the pressure force and the shearing stress. From the no-slip condition on the surface of the cylinder, we have from (1)

$$\mathrm{grad}P=-\frac{\partial}{\partial x}\boldsymbol{u}+\frac{1}{Re}\nabla^2\boldsymbol{u}\quad\text{on}\quad S. \tag{91}$$

The coordinate system taken in (91) is the absolute one, so that the first term results from the motion of the cylinder. From this equation, the pressure on the surface of the cylinder is given by

$$P=-u-\int\zeta\mathrm{d}y+\frac{1}{Re}\int\left(-\frac{\partial\zeta}{\partial y}\mathrm{d}x+\frac{\partial\zeta}{\partial x}\mathrm{d}y\right). \tag{92}$$

The drag coefficient due to the pressure force, C_{Dp}, is given by using integration by parts:

$$C_{D_p} = -2\int_0^{2\pi} \frac{\partial P}{\partial \theta} r_s \sin\theta \mathrm{d}\theta. \tag{93}$$

The drag coefficient due to the shearing stress on the surface of the cylinder, C_{D_f}, is given by

$$C_{D_f} = -\frac{2}{Re}\oint_S \zeta \mathrm{d}x. \tag{94}$$

Thus, we have the total drag force coefficient, C_D, by summing the above two coefficients.

For a circular cylinder, we easily find from (92)–(94) that

$$C_D = \frac{2}{Re}\int_0^{2\pi}\left(\zeta - \frac{\partial \zeta}{\partial r}\right)\sin\theta \mathrm{d}\theta. \tag{95}$$

Since the vorticity given in the absolute coordinate system is the same as in the relative coordinate system, the composite solution of $\tilde{\zeta}$ near the surface of the cylinder in the Laplace transform plane is given from the solutions in regions (I), (II) and (V) by

$$\tilde{\zeta} = 4\left(\frac{\varepsilon}{2p}\right)^{1/2}\frac{K_1\left((2\varepsilon p)^{1/2} r\right)}{K_0\left((2\varepsilon p)^{1/2}\right)}\sin\theta + \tilde{C}\frac{1}{r}\sin\theta \tag{96}$$

where $\tilde{C}$ is defined by

$$\tilde{C} = \frac{2}{p}\left[\frac{1}{\gamma + \frac{1}{2}\log p + \frac{1}{2}\log\varepsilon - \frac{1}{2}\log 2} - \frac{1}{\gamma + \log\frac{1}{2} + \log\varepsilon + \frac{1}{2}\log\left(\frac{2p+\varepsilon}{\varepsilon}\right) - \frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\left(\frac{2p}{\varepsilon + 2p}\right) + 1\right\}}\right]. \tag{97}$$

Therefore, we have the Laplace transform of the drag coefficient, $\tilde{C_D}$:

$$\tilde{C_D} = \frac{4\pi}{Re}\left[\varepsilon\left\{1 + \frac{K_2\left((2\varepsilon p)^{1/2}\right)}{K_0\left((2\varepsilon p)^{1/2}\right)}\right\} + 2\left(\frac{\varepsilon}{2p}\right)^{1/2}\frac{K_1\left((2\varepsilon p)^{1/2}\right)}{K_0\left((2\varepsilon p)^{1/2}\right)} + \tilde{C}\right].$$

Carrying out the inverse Laplace transform, we finally have

$$C_D = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\exp(pt)\tilde{C_D}\mathrm{d}p \cong 2\pi\left[2\delta(t) + \frac{2}{\varepsilon}(\hat{I}_2 - \hat{I}_3) + 4\hat{I}_1\right] \tag{98}$$

where

$$\hat{I}_1 = \frac{2}{\pi^2}\frac{2}{Re}\int_0^\infty \exp\left(-\frac{x^2}{Re}t\right)\left[\frac{1}{x}\left(\frac{1}{J_0^2(x) + Y_0^2(x)} - \frac{1}{1 + \frac{4}{\pi^2}\left(\log\frac{1}{2}x + \gamma\right)^2}\right) - \frac{\pi}{2}\right]\mathrm{d}x$$
$$+ \frac{1}{(\pi t Re)^{1/2}} + \frac{2}{\pi^2}\frac{2}{Re}\int_0^\infty \exp\left(-\frac{x^2}{Re}t\right)\frac{1}{x}\frac{1}{1 + (4/\pi^2)\left(\log\frac{1}{2}x + \gamma\right)^2}\mathrm{d}x,$$

$$\hat{I}_2 = \frac{4}{\pi^2}\int_0^\infty \frac{1-\exp\left(-(t/Re)x^2\right)}{x\left[1+(4/\pi^2)\left(\gamma+\log\frac{1}{2}x\right)^2\right]}\mathrm{d}x,$$

$$\hat{I}_3 = \int_{Re/2}^\infty \frac{1}{x}\left(1-\exp\left(-\frac{t}{Re}x^2\right)\right)$$
$$\times\frac{1}{\left[\gamma+\frac{1}{2}\log\frac{1}{4}\left(x^2-(\frac{1}{2}Re)^2\right)-\frac{x^2}{2(Re/2)^2}\log\left(\frac{x^2}{x^2-(Re/2)^2}\right)+\frac{1}{2}\right]^2+\frac{1}{4}\pi^2}\mathrm{d}x$$
$$+\int_0^{Re/2}\frac{\left(\frac{1}{2}Re\right)^2}{x^3\left(\frac{\left(\frac{1}{2}Re\right)^2}{x^2}\right)^2}$$
$$\times\frac{1-\exp\left(-(t/Re)x^2\right)}{\left[\gamma+\frac{1}{2}\log\frac{1}{4}\left(\left(\frac{1}{2}Re\right)^2-x^2\right)+\frac{x^2}{2\left(Re/2\right)^2}\log\left(\frac{x^2}{\left(Re/2\right)^2-x^2}\right)-\frac{1}{2}\right]^2+\frac{\pi^2}{4}\left(\frac{x}{R_e/2}\right)^4}\mathrm{d}x.$$

To compare the present result with Bar-Lev & Yang (1975), we can obtain the asymptotic solution of C_D for $t \ll 1$, using the asymptotic expansions of the modified Bessel functions K_m. The final expression becomes

$$\tilde{C}_D \approx \varepsilon\frac{4\pi}{Re}\left(2+\frac{4}{x}+\frac{2}{x^2}-\frac{1}{2x^3}+\frac{\tilde{C}}{\varepsilon}\right) \tag{99}$$

where $x=(2\varepsilon p)^{1/2}$ and $\tilde{C}$ is given by (97). Thus, we finally have (see the Appendix)

$$C_D^* = 2\pi\left[2\delta(t)+\frac{4}{(\pi t Re)^{1/2}}+\frac{2}{Re}H(t)-\frac{t^{1/2}}{Re(\pi Re)^{1/2}}+\int_{-\log t}^\infty\frac{\exp(-x)}{x^2}\mathrm{d}x\right]. \tag{100}$$

We note that the first four terms in the square bracket of (100) of the present result agree well with Bar-Lev & Yang (1975). The details of the analysis in the present section are contained in the typescript in the editorial file.

6. Numerical results

Figure 4(*a–d*) shows the drag coefficient C_D. In this figure, the solid line and the broken line are the present results calculated from (98) and (100), respectively. We see that the drag coefficient C_D^* given from the asymptotic expression (100) is only valid for very early time from the motion. The asymptotic expression (100) is valid for $x \gg 1$, so that it is theoretically reasonable for $t \ll Re$, but it is almost valid for $t \leqslant 1$ in the case of $Re \geqslant 0.5$. We see from this figure that the drag coefficient decreases monotonically to the steady one with time. Figure 5 shows the drag coefficient calculated by the present method together with earlier results. The present ones are for $t = 1000$. This implies that the present results are almost steady flow. Earlier theoretical results are given by Kaplun (1957) and Kida & Take (1992*b*) and the experimental ones are given by Tritton (1959) for the steady flow. The present results, shown as dots, are the first approximation; the result of Kaplun (1957) is the second approximation with respect to $1/\log Re$ and the results given by Kida & Take (1992*b*) are the higher approximation; however, we see that the present solution is in

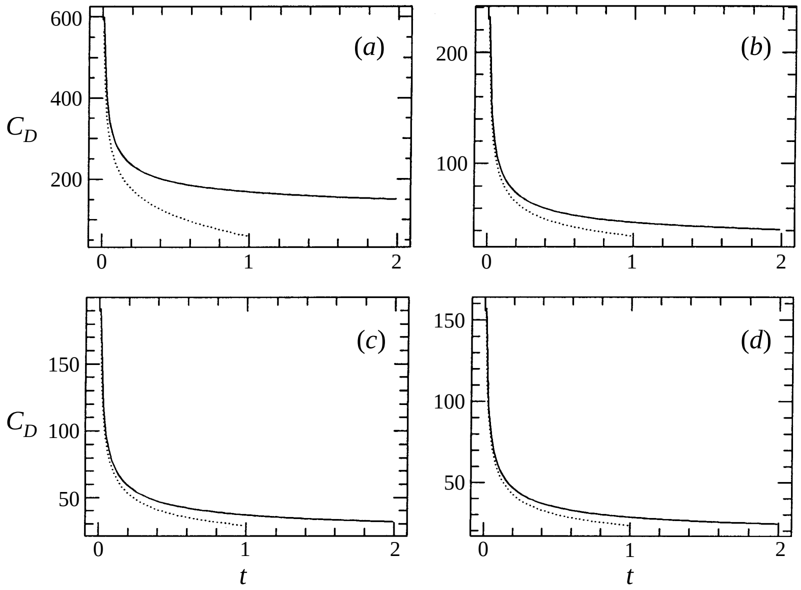

FIGURE 4. Drag coefficients C_D (solid line) and C_D^* (dotted line): (a) $Re = 0.1$, (b) $Re = 0.5$, (c) $Re = 0.7$, (d) $Re = 1.0$.

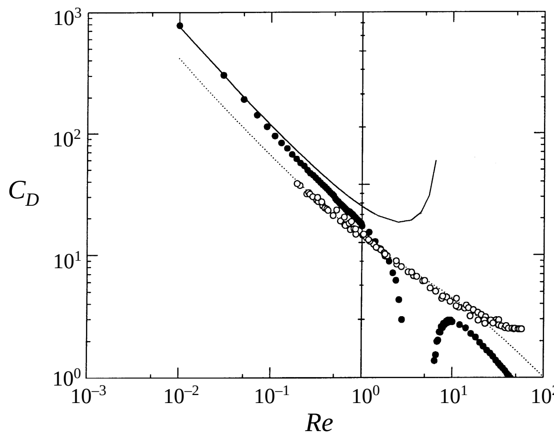

FIGURE 5. Comparison of the drag coefficient for steady flow with earlier theories and experiment: solid line is Kaplun (1957), dashed line is Kida & Take (1992b), open circles are experimental results given by Tritton (1975), and solid circles are the present results.

agreement with experimental works for steady flow for $Re \leqslant 1 \sim 2$. From (98), we can obtain the following relation for $t \to \infty$ and small Re:

$$C_D \sim -\frac{8\pi}{Re}\frac{1}{\gamma + \log \frac{1}{4}Re}.$$

This shows the increase of C_D with small Re. The details are in the typescript in the editorial files. The rapid change of the C_D curve near $Re = 3.7$ is due to the singularity of (98). In the second term of the right-hand side for $\hat{I}_3$ in (98), the integrand becomes very large near $\gamma + \log(R_e/4) - \frac{1}{2} \approx 0$: in this case the integrand is of $O(1/x^3 \log^2 x)$

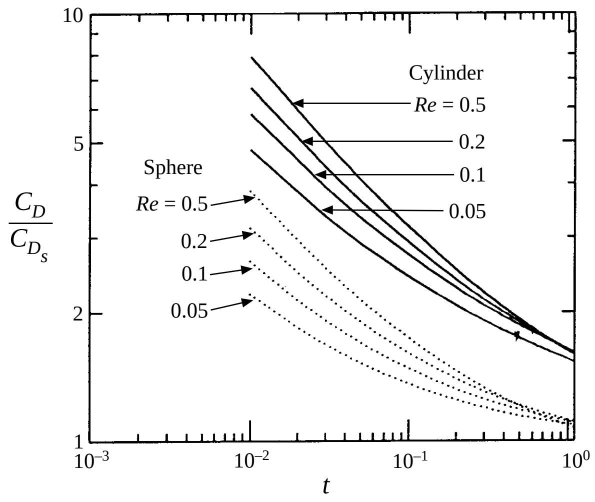

FIGURE 6. Ratio of the drag coefficient C_D and the steady one C_{D_S} for the circular cylinder and the sphere.

for $x \to 0$, so that $\hat{I}_3$ becomes very large with $Re \to 4\exp(\frac{1}{2} - \gamma)$, that is, C_D changes from positive to negative abruptly near this value of R_e.

Figure 6 shows a comparison with the case of a sphere given by Sano (1981). C_{Ds} is the drag coefficient in the steady flow. The drag coefficient decreases abruptly at the beginning of motion and decreases monotonically with time. This trend is the same as in the flow past the sphere, but its rate is much slower. We see for small t that $C_D/C_{Ds} \sim -(Re/(\pi t))^{1/2}/(\gamma + \log(Re/4))$ for a circular cylinder from (100) and the above relation. For the sphere, $C_D/C_{Ds} \sim (Re/(\pi t))^{1/2}$ is given by Sano (1981). Further, this figure shows that the three-dimensional flow becomes almost steady flow at dimensionless time unity, but it takes a long time to arrive at the steady flow in the two-dimensional case. This implies that unsteady motion in the two-dimensional flow is more important than that in the three-dimensional flow.

Figure 7(*a–c*) shows the time history of the streamline pattern around a circular cylinder in the case of $Re = 0.1$. The streamlines shown in this figures are obtained from (88), which is valid for $r = O(1)$. They are symmetric with respect to the x- and y-axes. The circulatory flow located in the immediate vicinity of the cylinder is seen from the very early stage of motion and migrates outwards along the y-axis with the progress of time. The flow near the surface of the cylinder from stagnancy is disturbed onwards, so that the fluid is pushed out at the front and is entrained at the rear, thus, the circulatory flow is formed. This feature is the same as in the three-dimensional flow (Bentwich & Miloh 1978). Figure 8 shows the velocity field at the same Reynolds number. The fluid at the front of the circular cylinder is pushed out and it is entrained at the rear, so the circulatory flow is formed.

Figure 9 shows the vorticity distribution on the surface of the cylinder. At the beginning of motion, the vorticity is large, $2Re^{1/2}/(\pi t)^{1/2}\sin\varphi$, which is obtained from (44), and decreases with time. Comparing with the result for the flow around the sphere (Sano 1981), $3Re^{1/2}/(2(\pi t)^{1/2})\sin\varphi$, we see that the vorticity around the cylinder is larger than around the sphere at the early stages of motion; however, the vorticity distributions for the steady flow are $2/(\log 4 - \gamma + (1/2) - \log Re)\sin\varphi$ for the cylinder (Skinner 1975) and $3/2\sin\varphi$ for the sphere (see Lamb 1945), hence its value

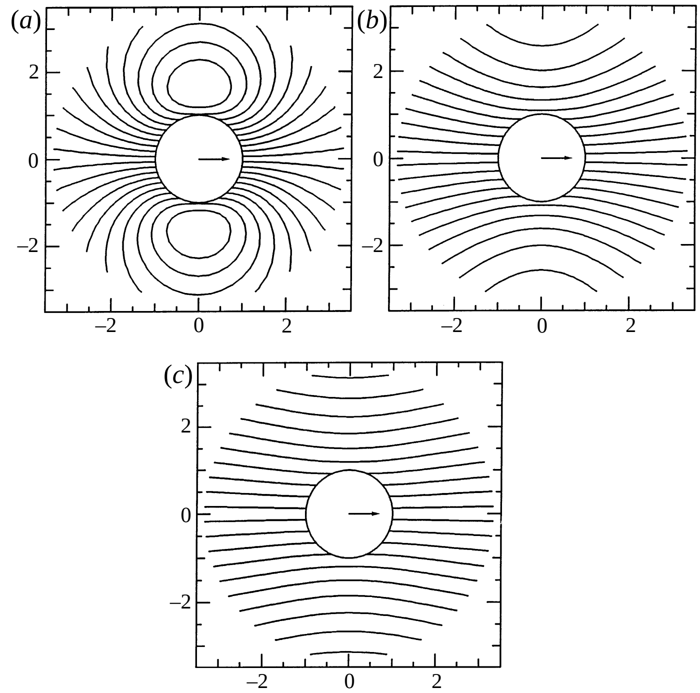

FIGURE 7. Time history of streamlines in the case $Re = 0.1$: (a) $t = 0.04$, (b) $t = 1$, (c) $t = 50$.

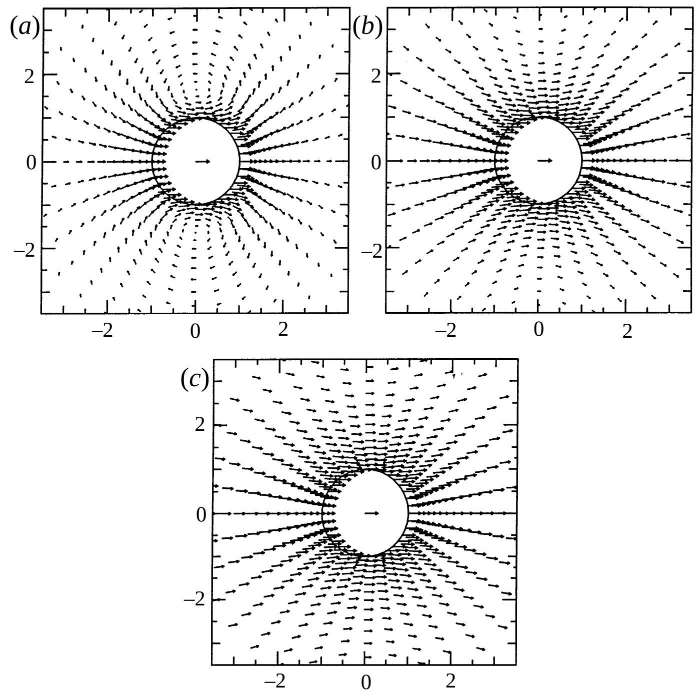

FIGURE 8. Time history of velocity vectors in the case $Re = 0.1$: (a) $t = 0.04$, (b) $t = 1$, (c) $t = 50$.

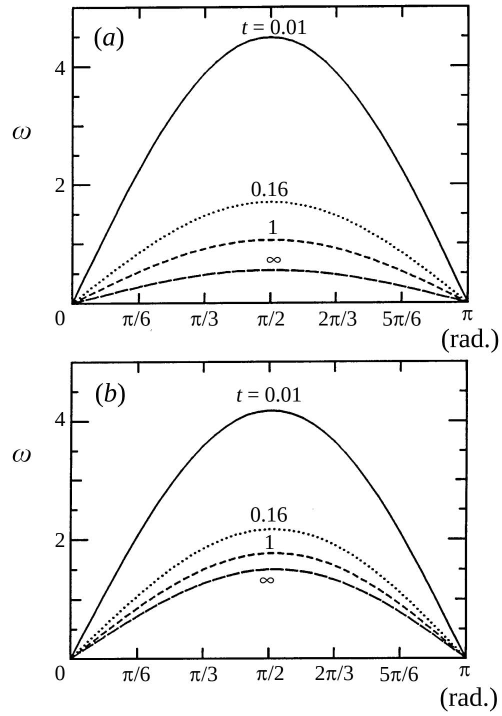

FIGURE 9. Vorticity distribution on the surfaces of (*a*) circular cylinder, (*b*) sphere.

is smaller in the flow past a circular cylinder than in the flow past a sphere, that is, the effect of unsteady motion on the flow is more important in the two-dimensional case.

Figure 10 shows the velocity profile u on the y-axis ($\theta = \pi/2$). This figure also implies the existence of the circulatory zone, that is, the reverse flow near the surface of the cylinder appears at earlier stages (see $t = 0.04$). At earlier stages, the reverse velocity is larger in the flow past a sphere than in the flow past a circular cylinder, as shown in this figure, so that the circulatory zone is larger in the former flow. From (43), we have for $p \gg 1$

$$\frac{\partial \tilde{\psi}}{\partial r} \approx \frac{1}{p}\left(\frac{2}{r^{1/2}}\exp\left(-(2\varepsilon p)^{1/2}(r-1)\right) - \frac{1}{r^2}\right)\sin\varphi.$$

Therefore, we have the velocity component of the φ-direction, u_φ for $t \ll 1$:

$$u_\varphi \approx -\left(\frac{2}{r^{1/2}}\mathrm{erfc}(\eta) - \frac{1}{r^2}\right)\sin\varphi$$

where

$$\eta = \left(\frac{Re}{4t}\right)^{1/2}(r-1).$$

From Bentwich & Miloh (1978), we have for the flow past a sphere:

$$u_\varphi \approx -\left(\frac{3}{2r}\mathrm{erfc}(\eta) - \frac{1}{2r^3}\right)\sin\varphi.$$

From these relations, the centre of the circulatory flow approximately becomes for

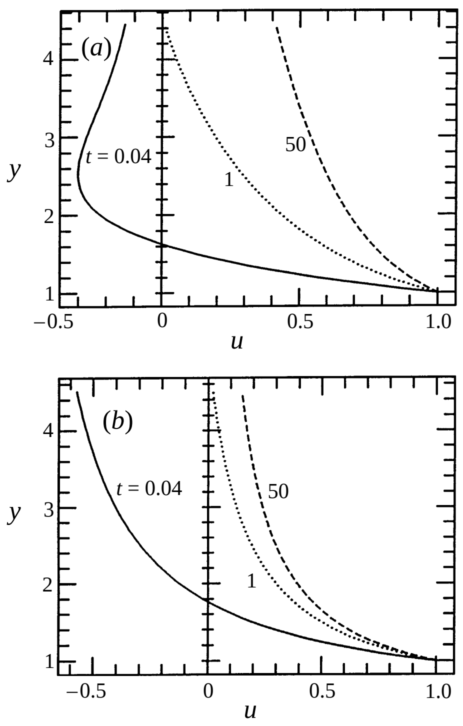

FIGURE 10. Velocity profile u on the y-axis: (a) circular cylinder, (b) sphere.

$t \ll 1$:

$$\frac{1}{3}\left(\frac{\pi}{Re}\right)^{1/2} t^{1/2} \text{ : sphere,}$$
$$\frac{1}{2}\left(\frac{\pi}{Re}\right)^{1/2} t^{1/2} \text{ : cylinder.}$$

Thus, the time development of the circulatory flow is found to be almost the same.

Figure 11(a–d) shows the vorticity distribution of flows past a circular cylinder and a sphere. In these figures, the steady flows are also shown. In the flow past a sphere, the vorticity distribution at $t = 0.04$ is very similar to the steady one, but in the flow past a circular cylinder it is different from the steady one. We, therefore, infer that the effect of unsteady motion of two-dimensional bodies on flow is more significant than of three-dimensional bodies.

7. Conclusion

The low Reynolds number flow around an impulsively started cylinder is solved by using the method of matched asymptotic expansions. The governing integral equations are derived and the method of matched asymptotic expansions proposed by Kida & Take (1992a, b) is applied to these equations. The most important results in the present paper are that (i) five regions are necessary to complete the matching process, while in the flow past a sphere only three regions are necessary, and (ii) the three-dimensional flow becomes almost steady flow at dimensionless time unity, but it takes a long time to arrive at the steady flow in the two-dimensional flow.

The streamlines, the vorticity distribution and the drag coefficient are shown and

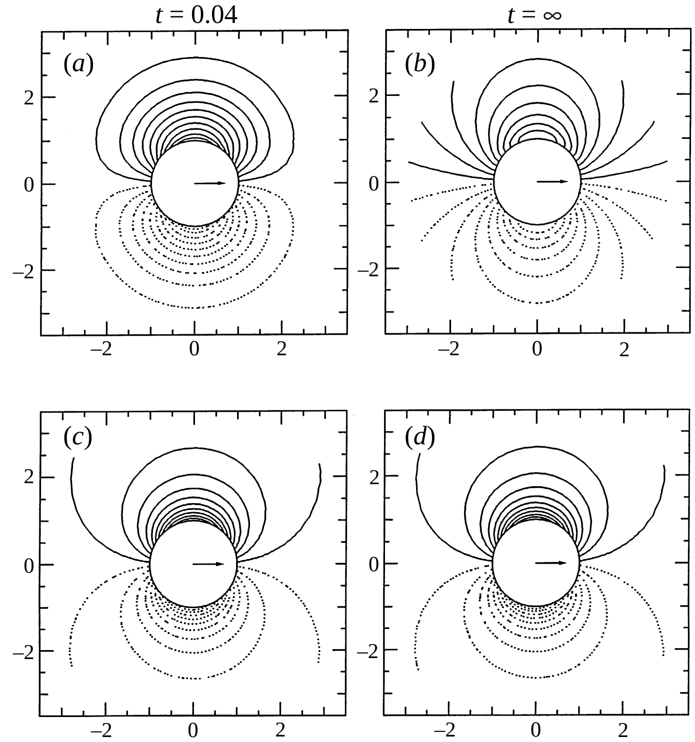

FIGURE 11. Equivorticity lines in the case $Re = 0.1$, $t = 0.04$ and $t = \infty$: (a) and (b) circular cylinder, (c) and (d) sphere.

compared with the flow around a sphere. Circulatory flow is formed and the time development of the global flow is almost the same as in the flow past a sphere. The drag coefficient decreases abruptly at the beginning of the and decreases monotonically with time. The rate of the decrease is less than that in the flow past a sphere. The vorticity field converges to its final steady form much more slowly in the case of the circular cylinder than for the sphere. The vorticity distribution on the surface of the circular cylinder is larger at the beginning of motion, decreases with time and finally becomes smaller than in the flow past a sphere.

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Appendix

In (98), we consider the case of $p \to \infty$. We use the following relation:

$$\frac{K_1(x)}{K_0(x)} \approx 1 + \frac{1}{2x} - \frac{1}{8x^2} + O(x^{-3}).$$

Then we have (99). Carrying out the Laplace inverse transformation, we have

$$C_D \approx \frac{4\pi}{R_e}\varepsilon\left(2\delta(t) + \frac{4}{(2\varepsilon)^{1/2}}\frac{1}{(\pi t)^{1/2}} + \frac{1}{\varepsilon}H(t) - \frac{1}{(2\varepsilon)^{3/2}}\frac{t^{1/2}}{\pi^{1/2}} + \frac{1}{\varepsilon}\mathscr{L}^{-1}(\tilde{C})\right)$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transformation.

For large p, $\tilde{C}$ becomes

$$\tilde{C} \approx \frac{\varepsilon}{4p^2}\frac{1}{\left(\gamma + \frac{1}{2}\log\frac{1}{2}\varepsilon p\right)^2}.$$

Here, we use the following relation:

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{1}{\gamma + \frac{1}{2}\log\frac{1}{2}\varepsilon p}\right)^2 &= \int_0^\infty \exp(-pt) \\ &\quad\times \frac{\gamma + \frac{1}{2}\log\frac{1}{2}\varepsilon p}{\left[\left(\gamma + \frac{1}{2}\log\frac{1}{2}\varepsilon p\right)^2 - \frac{1}{4}\pi^2\right]^2 + \pi^2\left(\gamma + \frac{1}{2}\log\frac{1}{2}\varepsilon p\right)^2}\mathrm{d}p \\ &\approx \frac{1}{t}\int_0^\infty \frac{\exp(-x)}{(-\log t)^2}\mathrm{d}x \approx \frac{8}{t(-\log t)^3}.\end{aligned}$$

Thus, we have

$$\mathscr{L}^{-1}(\tilde{C}) \approx \int_0^t \int_0^t \frac{8}{t(-\log t)^3}\mathrm{d}t\mathrm{d}t \approx 4\int_{-\log x}^\infty \frac{\exp(-x)}{x^2}\mathrm{d}x.$$

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